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Convexity and Finite Quantum Logics

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The notion of a superposition of a set of states and that of a Jauch–Piron state are geometrically interpreted in the context of the facial structure of the state space of a finite quantum logic.

1. INTRODUCTION

The state space $\Omega(L)$ of a finite quantum logic (orthomodular poset) (Beltrametti and Cassinelli, 1981) L is a polytope. Whenever L is classical, i.e., a Boolean lattice, then $\Omega(L)$ is the simplest kind of a polytope, namely a simplex. As geometrical objects, convex polytopes, or simply polytopes, have attracted the interest of many a mathematician. A considerable amount of literature has accumulated over the past 50 years which is concerned with the facial structure of polytopes. The collection of faces of a polytope, when ordered by set inclusion, forms a lattice. Notice that the face lattice of a polytope is Boolean if and only if the polytope is a simplex.

It is the purpose of this paper to give a geometrical meaning to both the notion of a superposition of states and to that of a Jauch–Piron state thereby relating them to the facial structure of the state space of the finite quantum logic.

2. PRELIMINARIES

Let V be a real vector space. Let C be a convex subset of V. A subset F of C is said to be a *face of* C if, for elements x, y in C and real number t in $[0, 1]$,

$$tx + (1 - t)y \in F \Leftrightarrow x, y \in F$$

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In particular, a face of C is a convex set. An element x in C is called an *extreme point of* C if the singleton set $\{x\}$ is a face of C. The empty subset of C and C itself are faces of C. Therefore, when ordered by set-inclusion, the collection $\mathscr{F}(C)$ of faces of C forms a complete lattice. A co-atom in the lattice $\mathscr{F}(C)$ is said to be a *facet* of C. Let M be a subset of C. The intersection of all faces of C which contain M is called the *face generated by* M and denoted by face(M). We write face(x) instead of face($\{x\}$) for an element x in C.

A subset P of the real vector space V which is the convex hull of a finite set is called a *polytope* (Brøndsted, 1983; Grünbaum, 1967). Notice that a polytope is convex and compact in the unique linear Hausdorff topology τ of the finite-dimensional subspace lin(P) of V. A polytope has finitely many extreme points and coincides with the convex hull of its extreme points. Let P°_r} be the relative topological interior of P, i.e., the interior with respect to the topology τ restricted to the affine span of P. If P is not empty, then P°_r} is not empty and

$$P^{\circ_r} = \{x \in P\colon \text{face}(x) = P\}$$

A face of a polytope is a polytope in its own right. The finite lattice $\mathscr{F}(P)$ of faces of a polytope P has interesting properties (Bennett, 1977). The atoms of $\mathscr{F}(P)$ are precisely the one point sets formed by the extreme points of P. Every face F different from P is contained in a facet and F coincides with the intersection of all facets which contain F.

Let L be a quantum logic, i.e., an orthomodular poset. Let $\Omega(L)$ be its state space, a convex subset of the real vector space $\mathbb{R}^L$. A state on L is called *pure* if it is an extreme point of $\Omega(L)$. If the orthomodular poset L is finite, then the state space $\Omega(L)$ is a polytope (Rüttimann, 1977).

A subset Δ of $\Omega(L)$ is said to be *unital* if for every nonzero element p in L there exists an element μ in Δ such that $\mu(p)$ equals one. The subset Δ is said to be *strong* if, for elements p, q in L,

$$\{\mu \in \Delta\colon \mu(p) = 1\} \subseteq \{\mu \in \Delta\colon \mu(q) = 1\} \;\Rightarrow\; p \le q$$

A state μ on L is said to be *Jauch–Piron* if, for elements p, q in L,

$$\mu(p) = 1 = \mu(q) \;\Rightarrow\; \exists r \le p, q \text{ with } \mu(r) = 1$$

Clearly, if L is a lattice, then a state is Jauch–Piron if and only if, for elements p, q in L,

$$\mu(p) = 1 = \mu(q) \;\Rightarrow\; \mu(p \wedge q) = 1$$

The collection of Jauch–Piron states is denoted by $\Omega_{JP}(L)$. If L is finite, then to every Jauch–Piron state μ there exists a unique element p_μ in L such that

$$\{p \in L: \mu(p) = 1\} = [p_\mu, 1]$$

p_μ is called the *support* of μ.

Let L be a quantum logic and let Δ be a subset of its state space $\Omega(L)$. A state μ is called a *superposition* of the states in Δ (Kläy, 1987; Varadarajan, 1968) if, for elements p in L,

$$\nu(p) = 1, \quad \forall \nu \in \Delta \;\Rightarrow\; \mu(p) = 1$$

$\mathrm{spp}(\Delta)$ denotes the collection of superpositions of the states in Δ. We write $\mathrm{spp}(\mu)$ instead of $\mathrm{spp}(\{\mu\})$ for an element μ in $\Omega(L)$. It is easily verified that $\mathrm{spp}(\Delta)$ is a face of the convex set $\Omega(L)$. Moreover, for subsets Δ, Δ_1, Δ_2 of $\Omega(L)$, (i) $\Delta \subseteq \mathrm{spp}(\Delta)$, (ii) $\Delta_1 \subseteq \Delta_2 \Rightarrow \mathrm{spp}(\Delta_1) \subseteq \mathrm{spp}(\Delta_2)$ and (iii) $\mathrm{spp}(\mathrm{spp}\,\Delta) = \mathrm{spp}(\Delta)$.

Let p be an element in L and define $a(p)$ to be the set of all states μ such that $\mu(p)$ equals one. It follows that the set $a(p)$ is a face of $\Omega(L)$. If $p \leq q$, then $a(p) \subseteq a(q)$.

For details concerning orthomodular posets, measures and states on such structures the reader may consult Birkhoff (1967), Kalmbach (1983), Rüttimann and Schindler (1987), and Rüttimann (1989, 1990, 1992).

3. FACES AND JAUCH–PIRON STATES

The following result establishes a relationship between superpositions and the facial structure of the state space of a finite quantum logic.

Theorem 3.1. Let L be a finite orthomodular poset. Let Δ be a subset of the state space $\Omega(L)$ of L. Then

$$\mathrm{spp}(\Delta) = \mathrm{face}(\Delta)$$

Proof. Since $\mathrm{spp}(\Delta)$ is a face of $\Omega(L)$, it follows that

$$\Delta \subseteq \mathrm{face}(\Delta) \subseteq \mathrm{spp}(\Delta) \subseteq \Omega(L)$$

If $\mathrm{face}(\Delta)$ coincides with $\Omega(L)$, we are done. Suppose now that $\mathrm{face}(\Delta)$ is a proper subset of $\Omega(L)$. Let F be a facet of $\Omega(L)$ which contains $\mathrm{face}(\Delta)$. By Rüttimann (1977), Theorem 4.2, F is equal to $a(p)$ for some element p in L. Then Δ is contained in the face $a(p)$. By definition, every superposition of Δ is contained in $a(p)$, i.e., $\mathrm{spp}(\Delta)$ is a subset of F. Therefore,

$$\mathrm{spp}(\Delta) \subseteq \bigcap \{F \subseteq \Omega(L): F \text{ facet of } \Omega(L);\ \mathrm{face}(\Delta) \subseteq F\} = \mathrm{face}(\Delta)$$

Let L be a finite quantum logic and let μ be a state on L. It follows, by Theorem 3.1, that

$$\mathrm{spp}(\mu) = \{\mu\}$$

if and only if μ is a pure state on L.

The following lemma admits extensions in various directions. It is presented here in the required form.

Lemma 3.2. Let L be a finite orthomodular poset.

(i) Let μ be a Jauch–Piron state and let the element p_μ be its support. Then

$$\mathrm{spp}(\mu) = a(p_\mu)$$

(ii) Suppose that the state space $\Omega(L)$ of L is strong. The state μ is Jauch–Piron if and only if there exists an element p in L such that

$$\mathrm{spp}(\mu) = a(p)$$

Proof. (i) Let ν be an element of the face $a(p_\mu)$. Let p be an element in L and suppose that $\mu(p)$ equals one. Then $p_\mu \leq p$ and it follows that $\nu(p)$ equals one. Therefore ν is a superposition of $\{\mu\}$. Conversely, let ν be a superposition of $\{\mu\}$. Since $\mu(p_\mu)$ equals one, we conclude that $\nu(p_\mu)$ equals one.

(ii) Let μ be a state and suppose that there exists an element p in L such that the condition is satisfied. Furthermore, assume that, for elements q and r,

$$\mu(q) = 1 = \mu(r)$$

Then μ belongs to the face $a(q) \cap a(r)$ and therefore, by Theorem 3.1,

$$a(p) = \mathrm{spp}(\mu) \subseteq a(q) \cap a(r)$$

Then $p \leq q, r$. Since μ belongs to $\mathrm{spp}(\mu)$, we conclude that $\mu(p)$ equals one. The converse follows from (i).

The following theorem gives us information about the geometrical structure of the set $\Omega_{\mathrm{JP}}(L)$ of all Jauch–Piron states on L.

Theorem 3.3. Let L be a finite orthomodular poset. Let $\Omega_{\mathrm{JP}}(L)$ be the collection of Jauch–Piron states on L. For each element p in L let $a(p)$ be the set of all states which evaluate to one on p. Let $a(p)^{\circ_r}$ be the relative topological interior of $a(p)$. Then

$$\Omega_{\mathrm{JP}}(L) \subseteq \bigcup_{p \in L} a(p)^{\circ_r}$$

Furthermore, if $\Omega(L)$ is strong, then

$$\Omega_{\text{JP}}(L) = \bigcup_{p \in L} a(p)^{\circ_r}$$

Proof. Let μ be a Jauch–Piron state. Then, by Theorem 3.1 and Lemma 3.2(i),

$$\text{face}(\mu) = \text{spp}(\mu) = a(p_\mu)$$

Since the face $a(p_\mu)$ is a polytope, we conclude that μ is an element in $a(p)^{\circ_r}$.

Suppose now that $\Omega(L)$ is strong. Let μ be an element in $a(p)^{\circ_r}$ for some element p in L. Then

$$\text{spp}(\mu) = \text{face}(\mu) = a(p)$$

By Lemma 3.2(ii), μ is Jauch–Piron.

Provided that the state space $\Omega(L)$ of the finite quantum logic L is strong, the following corollary shows that there are 'plenty' of Jauch–Piron states. More precisely, the relative topological interior of $\Omega(L)$ and the relative topological interior of each facet of $\Omega(L)$ belong to $\Omega_{\text{JP}}(L)$.

Corollary 3.4. Let L be a finite orthomodular poset and suppose that the state space $\Omega(L)$ is strong. Let μ be a state on L. If

$$\text{codim}(\text{face}(\mu)) \leq 1$$

then μ is Jauch–Piron.

Proof. If codim(face(μ)) equals zero, then face(μ) coincides with the face $a(1)$. By Theorem 3.1 and Lemma 3.2(ii), μ is Jauch–Piron.

If codim(face(μ)) equals one, then face(μ) is a facet of $\Omega(L)$. By Rüttimann (1977), Theorem 4.2, there exists an element p such that $a(p)$ coincides with face(μ). Again, the assertion follows, by Theorem 3.1 and Lemma 3.2(ii).

Theorem 3.5. Let L be a finite orthomodular poset. Let $\Omega(L)$ be its state space and let $\Omega_{\text{JP}}(L)$ be the collection of Jauch–Piron states. Then TAE:

(i) The set $\Omega(L)$ is strong.
(ii) The set $\Omega_{\text{JP}}(L)$ is unital.
(iii) The set $\Omega_{\text{JP}}(L)$ is strong.

Proof. (i) $\Rightarrow$ (ii): Let p be a nonzero element in L. Then the face $a(p)$ is non-empty and therefore, $\Omega(L)$ being a polytope, $a(p)^{\circ_r}$ is not empty.

Select an element μ in $a(p)^{\circ_r}$. Then $\mu(p)$ equals one and, by Theorem 3.3, μ is Jauch–Piron.

(ii) $\Rightarrow$ (iii): Let $\Omega_{JP}(L)$ be a unital set of states. Suppose that, for elements p, q in L, $a(p)$ is contained in $a(q)$. The orthomodular poset L is atomic, so let r be an atom with $r \leq p$. Then there exists an element μ in $\Omega_{JP}(L)$ such that $\mu(r)$ equals one. Consequently, $\mu(p)$ is equal to one and so is $\mu(q)$. Then there exists an element $s \leq r, q$ with $\mu(s)$ equal to one. Since r is an atom, s and r are equal, hence $r \leq q$. This holds true for all atoms majorized by p. Since L is also atomistic, we conclude that $p \leq q$.

(iii) $\Rightarrow$ (i): This is obvious.

4. CONVEXITY AND JAUCH–PIRON STATES

Lemma 4.1. Let L be an orthomodular lattice. The set $\Omega_{JP}(L)$ of Jauch–Piron states is convex.

Proof. Let μ and ν be Jauch–Piron states. Let ξ denote the convex combination $t\mu + (1 - t)\nu$, where $0 < t < 1$. If, for elements p, q in L,

$$\xi(p) = 1 = \xi(q)$$

then

$$\mu(p) = 1 = \nu(p) \qquad \text{and} \qquad \mu(q) = 1 = \nu(q)$$

Therefore,

$$\mu(p \wedge q) = 1 = \nu(p \wedge q)$$

which implies that $\xi(p \wedge q)$ equals one.

Theorem 4.2. Let L be a finite orthomodular poset. If L admits a convex unital set Δ of Jauch–Piron states, then L is a lattice.

Proof. Let $p_1, p_2, \ldots, p_n$ be atoms in L. Let, for $i = 1, 2, \ldots, n$, μ_i be an element in Δ such that $\mu_i(p_i)$ equals one. Notice that p_i coincides with p_{μ_i}. Let ν denote the state $n^{-1} \sum_{i=1}^{n} \mu_i$. By hypothesis, ν is an element in Δ. Then

$$1 = \nu(p_\nu) = \frac{1}{n} \sum_{i=1}^{n} \mu_i(p_\nu)$$

and it follows that, for $i = 1, 2, \ldots, n$, $\mu_i(p_\nu)$ equals one. Therefore p_ν is an upper bound of the set $\{p_1, p_2, \ldots, p_n\}$. Let r be an upper bound for $\{p_1, p_2, \ldots, p_n\}$. Since $p_{\mu_i} \leq r$, for $i = 1, 2, \ldots, n$, we conclude that $\nu(r)$ equals one and consequently $p_\nu \leq r$. Therefore the supremum of every subset consisting of atoms exists in L. Since L is atomic and atomistic, we conclude, by the generalized associative law, that L is a lattice.

Let us close this paper with the following observation.

An orthomodular poset L is said to have the *Jauch–Piron property* if every state on L is Jauch–Piron.

Theorem 4.3. Let L be a finite orthomodular lattice. Its set of states $\Omega(L)$ is unital and L has the Jauch–Piron property if and only if L is a Boolean lattice.

Proof. See Rüttimann (1977), Theorem 4.3.

This theorem together with Theorem 4.2, yields as an immediate corollary the following result:

Corollary 4.4. Let L be a finite orthomodular poset. Its set of states $\Omega(L)$ is unital and L has the Jauch–Piron property if and only if L is a Boolean lattice.

Proof. Notice that $\Omega_{JP}(L)$ is convex, since it coincides with $\Omega(L)$ and therefore, by Theorem 4.2, L is a lattice. The assertion now follows, by Theorem 4.3.

We remark that this result was obtained in Bunce *et al.* (1985), partially relying on methods used in Rüttimann (1977).

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